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Straight-line path approximation for high energy elastic and inelastic scattering in quantum gravity

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Abstract. The asymptotic behavior of Planck energy elastic and inelastic amplitudes in quantum gravity is studied by means of the functional integration method. A straight-line path approximation is used to calculate the functional integrals which arise. Closed relativistically invariant expressions are obtained for the two "nucleons" elastic and inelastic amplitudes including the radiative corrections. Under the requirement of "softness" of the secondary gravitons a Poisson distribution for the number of particles emitted in the collision is found.

1 Introduction

Planck energy gravitational scattering has received considerable attention in recent years because of its relation to fundamental problems like the strong gravitational forces near black holes, a string modification of the theory of gravity and some other effects of quantum gravity [1–14]. In a previous work [14] we have developed a method for constructing a scattering amplitude in quantum gravity by means of a functional integral used effectively in quantum electrodynamics [16,17,34,19–24].

A straight-line path approximation is formulated that can be used effectively to calculate the functional integrals that occur. It is shown that in the limit of asymptotically high energy $s \gg M_{\mathrm{PL}}^2 \gg t$, where M_{PL} is the Planck mass, at fixed momentum transfer t the elastic scattering amplitude of two "nucleons" has the form of a Glauber representation [14] with an eikonal function depending on the energy. A similar result is obtained by the "shock-wave method" proposed by 't Hooft [1], and by the method of effective topological theory in the Planck limit proposed by Verlinde and Verlinde [5] and by the summing of Feynman diagrams in the eikonal approximation [6]. The main advantage of the proposed approach over the others is the possibility of performing calculations in compact form and the correct structure of the Green's function and amplitudes etc. is not destroyed by approximations in the process of the calculations. In the present report we would like to apply the above method to study multiple bremsstrahlumg soft gravitons in collisions which are well known to be an important phenomenon in high energy particle collisions physics [25–27]. This problem has recently seen a renewal of interest in the context of the gravitational production of particles in an expanding universe [28]. This letter is organized as follows. In Sect. 2 we determine the elastic scattering amplitude of two particles in terms of the functional integral, remove divergences by the mass renormalization of the scattered "nucleons", and then, using the straight-line path approximation, we calculate the contributions of the radiative corrections to the Planck energy scattering amplitude. In Sect. 3 the problem of the multiple production of "soft" gravitons in high energy two "nucleon" collisions is intepreted by analogy with the bremsstrahlung emission of "soft" particles in electrodynamics; the inelastic scattering amplitude can be obtained by generalizing the procedure presented in Sect. 2. In Sect. 4 we consider the differential cross section of inelastic processes, and investigate the behavior of the distribution of secondary gravitons produced in high energy "nucleon" collisions. Finally in Sect. 5, we draw our conclusions.

2 Elastic scattering amplitudes

We consider the scattering of two scalar particles of the field $\varphi(x)$, a "nucleon" at high energies, at fixed transfer in quantum gravity. To construct the representation of the elastic scattering amplitude in the framework of the functional approach we first find the Green's function of the two "nucleons" case, then we must go over in the Green's function obtained to the mass shell respectively to the external ends of the "nucleon" line. Therefore, using the method of variational derivatives we shall determine the elastic scattering amplitude

$$\mathrm{i}(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2)T(p_1, p_2; q_1, q_2)$$

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$$
\begin{aligned}
&= \lim_{p_i^2, q_i^2 \to m^2} \left(\prod_{i=1,2} (q_i^2 - m^2)(p_i^2 - m^2) \right. \\
&\times \left. \int \mathrm{d}^4 x_i \mathrm{d}^4 y_i \mathrm{e}^{\mathrm{i}}(p_i x_i - q_i y_i) \right) \\
&\times \left(\exp \left[\frac{\mathrm{i}}{2} \int \mathrm{d}^4 \xi_1 \mathrm{d}^4 \xi_2 \frac{\delta}{\delta h^{\alpha\beta}(\xi_1)} \right. \right. \\
&\times D^{\alpha\beta\gamma\delta}(\xi_1 - \xi_2) \left. \frac{\delta}{\delta h^{\gamma\delta}(\xi_2)} \right] .G(x_1, y_1|h) \\
&\times G(x_2, y_2|h) S_0(h)) |_{h=0} \,, \qquad (2.1)
\end{aligned}
$$

where $G(x, y|h^{\mu\nu})$ is the Green's function of the "nucleon" in an external linearized gravitational field. Note that for the gravitational field in the first-order formalism one can write down an exact interaction Lagrangian that contains only a single vertex [14],

$$L(x) = L_{0,\varphi}(x) + L_{0,\mathrm{grav.}}(x) + L_{\mathrm{int}}(x),$$

where

$$L_{0\varphi}(x) = \frac{1}{2}[\partial^\mu \varphi(x) \partial_\mu \varphi(x) - m^2 \varphi^2(x)],$$

$$L_{\mathrm{int}}(x) = -\frac{\kappa}{2} h^{\mu\nu}(x) T_{\mu\nu}(x),$$

and $T_{\mu\nu}(x) = \partial_\mu \varphi(x) \partial_\nu \varphi(x) (1/2) \eta_{\mu\nu} [\partial^\sigma \varphi(x) \partial_\sigma \varphi(x) - m^2 \varphi^2(x)]$ is the energy-momentum tensor of the scalar field $\varphi(x)$.

The quantity

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}$$

in the form of functional integrals was found in [14]. Now,

$$
\begin{aligned}
G(x, y|h^{\mu\nu}) &= \mathrm{i} \int_0^\infty \mathrm{d}\tau \mathrm{e}^{-\mathrm{i}m^2\tau} \\
&\times \int [\delta^4 \nu]_0^\tau \exp \left(\mathrm{i}\kappa \int_0^\tau J_{\mu\nu} h^{\mu\nu} \right) \\
&\times \delta^4 \left(x - y - 2 \int_0^\tau \nu(\eta) \mathrm{d}\eta \right). \qquad (2.2)
\end{aligned}
$$

The coupling constant κ is related to Newton's constant of gravitation G by $\kappa^2 = 16\pi G$. In (2.2) we use the notation $\int J_i h = \int h^{\mu\nu}(z) J_{\mu\nu}(z)$ $(i = 1, 2)$, and $J_{\mu\nu}(z)$ is the current of the "nucleon" defined by

$$
\begin{aligned}
J_{\mu\nu}(z) &= \int_0^{\tau_i} \mathrm{d}\xi (\nu_\mu(\xi) \nu_\nu(\xi)) \\
&\times \delta \left(z - x_i + 2p_i \xi + 2 \int_0^\xi \nu_i(\eta) \mathrm{d}\eta \right), \qquad (2.3)
\end{aligned}
$$

and

$$[\delta^4 \nu]_{\tau_1}^{\tau^2} = \frac{\delta^4 \nu \exp[] - \mathrm{i} \int_{\tau_1}^{\tau^2} \nu_\mu^2(\eta) \prod_\eta \mathrm{d}^4\eta}{\int \delta^4 \nu \exp[] - \mathrm{i} \int_{\tau_1}^{\tau^2} \nu_\mu^2(\eta) \prod_\eta \mathrm{d}^4\eta}.$$

$[\delta^4 \nu]_{\tau_1}^{\tau^2}$ is a volume element of the functional space of the four dimensional functions $\nu(\eta)$ in the interval $\tau_1 \le \eta \le \tau_2$ and $S_0(h)$ is the vacuum expectation of the S-matrix in the external field $h_{\mu\nu}^{\mathrm{ext}}$. We shall henceforth disregard the contribution of the vacuum loops and put $S_0(h) = 1$. The function $D_{\alpha\beta\gamma\delta}(x)$ is the propagator of the free graviton field,

$$D_{\alpha\beta\gamma\delta}(x) = \omega_{\alpha\beta,\gamma\delta} \frac{\mathrm{i}}{(2\pi)^4} \int \frac{\mathrm{e}^{\mathrm{i}kx}}{k^2 - \mu^2 + \mathrm{i}\epsilon} \mathrm{d}^4 k, \qquad (2.4)$$

$$\omega_{\alpha\beta,\gamma\delta} = (\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}),$$

$$\eta_{\mu\nu} = (1, -1, -1, -1).$$

Substituting (2.2) to (2.1) and making a number of substitutions of the functional variables [14], we obtain a closed expression for the two-particle scattering amplitude in the form of functional integrals:

$$
\begin{aligned}
&T(p_1, p_2; q_1, q_2) = (\kappa^2) \int \mathrm{d}^4 x_1 \mathrm{d}^4 x_2 \mathrm{e}^{\mathrm{i}(p_1 - q_1)x_1 + \mathrm{i}(p_2 - q_2)x_2} \\
&\times D^{\alpha\beta\gamma\delta}(x) \int [\delta^4 \nu_1]_{-\infty}^\infty \int [\delta^4 \nu_2]_{-\infty}^\infty \\
&\times [p_1 + q_1 + 2\nu_1(0)]_\alpha [p_1 + q_1 + 2\nu_1(0)]_\beta \\
&\times [p_2 + q_2 + 2\nu_2(0)]_\gamma [p_2 + q_2 + 2\nu_2(0)]_\delta \, [] \\
&\times \int_0^1 \mathrm{d}\lambda \exp \left\{ \frac{1}{2} \mathrm{i}\kappa^2 \left[2\mathrm{i}\lambda \mathrm{e}^{\mathrm{i}kx} \int J_1 D J_2 \right. \right. \\
&\times \left. \left. \sum_{i=1,2} \left(\int \mathrm{d}^4 k J_i D J_i - \mathrm{i} \int_{-\infty}^\infty \delta_i m^2 \mathrm{d}\xi \right) \right] \right\}, \qquad (2.5)
\end{aligned}
$$

where the quantity $J_i^{\mu\nu}(k; p_i, q_i|\nu_i)$ is a conserving transition current and is given by

$$
\begin{aligned}
J_i^{\mu\nu}(k; p_i, q_i|\nu_i) &= 4 \int_{-\infty}^\infty \mathrm{d}\xi [p_i \theta(\xi) + q_i \theta(-\xi) + \nu(\xi)]^\mu \\
&\times [p_i \theta(\xi) + q_i \theta(-\xi) + \nu(\xi)]^\nu \\
&\times \exp \left(2\mathrm{i}k \left[p_i \xi_i \theta(\xi) + q_i \xi_i \theta(-\xi) \right. \right. \\
&+ \left. \left. \int_0^\xi \nu_i(\eta) \mathrm{d}\eta \right] \right), \qquad (2.6)
\end{aligned}
$$

$$J_i.D.J_k = \int\int \mathrm{d}z_1 \mathrm{d}z_2 J_i^{\mu\nu}(z_1) D_{\mu\nu\alpha\sigma}(z_1 - z_2) J_k^{\alpha\sigma}(z_2);$$
$$i, k = 1, 2.$$

The scattering amplitude (2.5) is interpreted as the residue of the two-particle Green's function (2.1) at the poles corresponding to the "nucleon" ends. The factor of the type $\exp\left(-(\mathrm{i}\kappa^2/2) \sum_{i=1,2} \int J_i D J_i\right)$ of (2.5) takes into account the radiative corrections to the scattered nucleons, while $\exp\left(\mathrm{i}\kappa^2 \mathrm{i}\lambda \mathrm{e}^{\mathrm{i}kx} \int J_1 D J_2\right)$ describes virtual-graviton exchange among them. The integral with respect to dλ ensures subtraction of the contribution of the freely propagating particles from the matrix element. The functional variables $\nu_1(\eta)$ and $\nu_2(\eta)$, formally introduced for obtaining the solution for the Green's function, describe the deviation of a particle trajectory from the straight-line paths. The functional with respect to $[\delta^4\nu_i]$ $(i = 1,2)$ corresponds to summation over all possible trajectories of the colliding particles. Expanding the expression (2.5) with respect to the coupling constant κ^2 and taking the functional integrals with $\nu_i(\eta)$, we obtain the well-known series of usual perturbation theory for two-particle scattering. From the consideration of the integrals over ξ_1 and ξ_2 for $\exp\left(-(\mathrm{i}\kappa^2/2)\sum_{i=1,2}\int J_i D J_i\right)$ it is seen that the radiative corrections result in divergent expressions of the type $\delta_i m^2 \times (A \to \infty)$. To regularize them, it is necessary to renormalize the mass, that is, to separate from $\exp\left(-(\mathrm{i}\kappa^2/2)\sum_{i=1,2}\int J_i D J_i\right)$ the terms $\delta_i m^2 \times (A \to \infty)$; $i = 1,2$, after which we go over in (2.5) to the observed masses $m_{iR}^2 = m_{i0}^2 + \delta_i m^2$.

Hitherto, no assumptions have been made. To advance in the investigation of the elastic amplitude we make the following assumption. We assume that all gravitons are "soft", i.e. their four-momenta are small compored with the momentum of the two "nucleon" system as well as the momentum between them and satifies the following condition:

$$\frac{1}{\sqrt{s}}\sum_{i=1}^{N} k_{0i} \ll 1;$$
$$\left|\sum_{i=1}^{N} k_{i\perp}\right| \ll |p_{1\perp} - q_{1\perp}| \approx |p_{2\perp} - q_{2\perp}|, \qquad (2.7)$$

where the particle momentum components are given in the centre of mass system, the moment of the intial "nucleons" being taken along the z axis. This means that in the propagators we can neglect terms of the form $\sum_{i\neq j} k_i k_j$ compared with $2p\sum_i k_i$, i.e. we can make the substitution

$$\left[m^2 - \left(p - \sum_{i=1}^{n} k_i\right)^2\right]^{-1} \to \left[2p\sum_{i=1}^{n} k_i - \sum_{i=1}^{n} k_i^2\right]^{-1},$$

where p is the momentum of one of the "nucleons" and k_i are the momenta of the gravitons. This approximation, which is called the straight-line path approximation, corresponds [14–17,34,19–21] to the approximate calculation of the Feynman path integrals in (2.5) in accordance with the rule

$$\int [\delta^4\nu] F_1[\nu] \exp\{F_2[\nu]\} = \overline{F_1[\nu]} \exp\left\{\overline{F_2[\nu]}\right\},) \qquad (2.8)$$

$$\overline{F_i[\nu]} = \int [\delta^4\nu] F_i[\nu]; \quad i = 1,2.$$

In this approximation, (2.8), the scattering amplitude of the elastic process takes the form

$$T(p_1,p_2;q_1,q_2) = \kappa^2 R(t) \int \mathrm{d}^4 x \mathrm{e}^{\mathrm{i}(p_1-q_1)x}$$
$$\times \Delta(x;p_1,p_2;q_1,q_2) \int_0^1 \mathrm{d}\lambda$$
$$\times \exp\{\mathrm{i}\lambda\chi(x;p_1,p_2,q_1,q_2)\}, \qquad (2.9)$$

where

$$\Delta(x;p_1,p_2;q_1,q_2) = \int \mathrm{d}^4 k D^{\mu\nu\rho\sigma}(k) \exp[\mathrm{i}kx]$$
$$\times [k+p_1+q_1]_\mu [k+p_1+q_1]_\nu$$
$$\times [-k+p_2+q_2]_\rho [-k+p_2+q_2]_\sigma, \qquad (2.10)$$

$$\overline{J_i^{\mu\nu}(k,p_i,q_i)} = \int [\delta^4\nu_i]_{-\infty}^{\infty} J_i^{\mu\nu}(k,p_i,q_i|\nu_i)$$
$$= \left[\frac{(2p_i+k)^\mu (2p_i+k)^\nu}{2p_i k + k^2 + \mathrm{i}\epsilon} - \frac{(2q_i-k)^\mu (2q_i-k)^\nu}{2q_1 k - k^2 - \mathrm{i}\epsilon}\right], \qquad (2.11)$$

$$\chi(x;p_1,p_2,q_1,q_2) = -\frac{\mathrm{i}\kappa^2}{(2\pi)^4}\int \mathrm{d}^4 k \mathrm{e}^{\mathrm{i}kx} D_{\mu\nu\rho\sigma}(k)$$
$$\times \overline{J_1^{\mu\nu}(-k,p_1,q_1) J_2^{\rho\sigma}(k,p_1,q_2)}, \qquad (2.12)$$

$$\overline{J_1^{\mu\nu}(-k,p_1,q_1) J_2^{\rho\sigma}(k,p_2,q_2)}$$
$$= \int [\delta^4\nu_1]_{-\infty}^{\infty} [\delta^4\nu_2]_{-\infty}^{\infty}$$
$$\times J_1^{\mu\nu}(-k,p_1,q_1|\nu_1) J_2^{\mu\nu}(k,p_2,q_2|\nu_2)$$
$$= \left[\frac{(2p_1+k)^\mu (2p_1+k)^\nu}{2p_1 k + k^2 + \mathrm{i}\epsilon} - \frac{(2q_1-k)^\mu (2q_1-k)^\nu}{2q_1 k - k^2 - \mathrm{i}\epsilon}\right]$$
$$\times \left[\frac{(2p_2-k)^\rho (2p_2-k)^\sigma}{2p_2 k - k^2 - \mathrm{i}\epsilon} - \frac{(2q_2+k)^\rho (2q_2+k)^\sigma}{2q_2 k + k^2 + \mathrm{i}\epsilon}\right], \qquad (2.13)$$

$$R(t) = \exp\left\{\sum_{i=1}^{2}\left[\frac{\mathrm{i}\kappa^2}{2(2\pi)^2}\int \mathrm{d}^4 k D_{\mu\nu\rho\sigma}(k) \times \overline{J_i^{\mu\nu}(k;p_i,q_i) J_i^{\rho\sigma}(-k;p_i,q_i)} - \delta_i m^2 (A\to\infty)\right]\right\}$$
$$= \exp\left\{\frac{\mathrm{i}\kappa^2}{2(2\pi)^2}\sum_{i=1}^{2}\int \mathrm{d}^4 k D_{\mu\nu\rho\sigma}(k) \times \left[\frac{(2p_i+k)^\mu (2p_i+k)^\nu (2p_i+k)^\rho (2p_i+k)^\sigma}{(2p_i k + k^2)^2} + \frac{(2q_i+k)^\mu (2q_i+k)^\nu (2q_i+k)^\rho (2q_i+k)^\sigma}{(2q_i k + k^2)^2} - \frac{2(2p_i+k)^\mu (2p_i+k)^\nu (2q_i+k)^\rho (2q_i+k)^\sigma}{(2p_i k + k^2)(2q_i k + k^2)}\right]\right\}. \qquad (2.14)$$

It is interesting to note that the contribution of the radiative corrections (2.14) can be factorized in the given approximation of (2.8) in the form of a factor $R(t)$. A similar factorization of the contributions of radiative corections occurs in the case of quantum electrodynamics [32]. In the calculation of $R(t)$ we must take care of the infrared divergences which we have treated above by the insertion of a small graviton mass μ. Evaluating the integrals in (2.14) for the radiative corrections (2.14) we obtain the following expression [15]:

$$R(t)_{t<0} = \exp\left\{ \frac{\kappa^2 m^2 t}{2(2\pi)^2} \left[\ln\frac{m^2}{\mu^2} - \frac{m^2}{\sqrt{-t(4m^2-t)}} \times \left(\ln\frac{m^2\sqrt{4m^2-t}}{\mu^2} \ln\frac{\sqrt{4m^2-t}+\sqrt{-t}}{\sqrt{4m^2-t}-\sqrt{-t}} \right) + \Phi(z_1) - \Phi(z_2) \right] \right\}, \quad (2.15)$$

$$\Phi(z) = \int_0^z \frac{\mathrm{d}y}{y} \ln|1-y|;$$

$$z_1 = \frac{\sqrt{4m^2-t}+\sqrt{t}}{2\sqrt{4m^2-t}};$$

$$z_1 = \frac{\sqrt{4m^2-t}-\sqrt{t}}{2\sqrt{4m^2-t}}.$$

Let us consider the asymptotic behavior of the scattering amplitude (2.9) at high energy $s \to \infty$ at fixed momentum transfer t (forward scattering). We make the calculation in the centre of mass system of the colliding particles: $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$, and we direct the z axis along the momentum $\mathbf{p}_1$. In the high energy limit $s \gg M_{\mathrm{PL}}^2 \gg t$ where M_{PL} is the Planck mass, at fixed momentum transfer t limited by the condition $|t| \ll m^2$, the values of the eikonal function and radiative corrections are

$$\chi(\mathbf{x}_\perp) = \frac{\kappa^2.s}{2\pi} K_0(\mu|\mathbf{x}_\perp),$$

$$R(t) = \exp(at),$$

where $K_0(\mu|\mathbf{x}_\perp)$ is the MacDonald function of zeroth order, and

$$a = \frac{2Gm^2}{\pi}\left(\ln\frac{m^2}{\mu^2}+\frac{1}{2}\right), \quad (2.16)$$

where μ is a graviton mass which serves as an infrared cutoff. Thus, in the given asymptotic limit the expression for the elastic scattering amplitude (2.8) has the form[1] [31]

$$T(s,t) = -2z\mathrm{i}(s-u)f(t)\mathrm{e}^{at}, \quad (2.17)$$

[1] Allowance for the identity of the "nucleons" leads to terms that vanish in the limit $s \to \infty$ and for t fixed when expression (2.17) is symmetrized.

where

$$f(t) = \frac{1}{2}\int \mathrm{d}^2\mathbf{x}_\perp \mathrm{e}^{-\mathrm{i}\mathbf{q}_\perp\mathbf{x}_\perp}(\mathrm{e}^{-\mathrm{i}\chi(\mathbf{x}_\perp)}-1) \quad (2.18)$$

is the elastic scattering amplitude without taking into account radiations corrections, and $t = -\mathbf{q}_\perp{}^2$. Formula (2.17) shows that allowance for radiative effects leads to a diffraction behavior of the high energy small-angle scattering amplitude. The forces due to the change of graviton between the "nucleons" obviously have a range $\hbar/\mu c$, it being assumed that $\hbar/\mu c \gg \kappa(\hbar/mc)$. Thus, in the region $\mu^2 \le |t| \le m^2$, allowance for graviton exchange becomes important and leads to an eikonal structure of $f(t)$.

3 Inelastic amplitudes

Here we shall consider a generalization of the above method to the construction of inelastic amplitudes. The production of secondary particles in the collision of two "nucleons" is intepreted by analogy with bremsstrahlung emission of soft particles in electrodynamics, i.e. the colliding "nucleons" interact by changing virtual quanta of the field $h^{\mu\nu}$ and emit at the same time secondary particles [19,25]. The amplitude of the above inelastic process can be obtained as follows. We first construct the scattering amplitude $T(p_1,p_2;q_1,q_2|h^{\mathrm{ext}})$ of the two "nucleons" in the presence of the external classical field $h_{\mu\nu}^{\mathrm{ext}}$. The quantity $T(p_1,p_2;q_1,q_2|h^{\mathrm{ext}})$ can be obtained by expression (2.1) in which one must set $h = h_{\mu\nu}^{\mathrm{ext}}$ after variational derivatives have been taken. As a result, we have

$$\begin{aligned}
&T(p_1,p_2;q_1,q_2|h^{\mathrm{ext}}) \\
&= (\kappa^2)\int \mathrm{d}^4x_1\mathrm{d}^4x_2\mathrm{e}^{\mathrm{i}(p_1-q_1)x_1+\mathrm{i}(p_2-q_2)x_2} \\
&\times D^{\alpha\beta\gamma\delta}(x)\int[\delta^4\nu_1]_{-\infty}^{\infty}\int[\delta^4\nu_2]_{-\infty}^{\infty} \\
&\times[p_1+q_1+2\nu_1(0)]_\alpha[p_1+q_1+2\nu_1(0)]_\beta \\
&\times[p_2+q_2+2\nu_2(0)]_\gamma[p_2+q_2+2\nu_2(0)]_\delta \\
&\times\int_0^1\mathrm{d}\lambda\exp\left\{\frac{1}{2}\mathrm{i}\kappa^2\left[2\mathrm{i}\lambda\mathrm{e}^{\mathrm{i}kx}\int J_1DJ_2\right.\right. \\
&\left.\left.\times\sum_{i=1,2}\left(\int\mathrm{d}^4kJ_iDJ_i-\mathrm{i}\int_{-\infty}^{\infty}\delta_i m^2\mathrm{d}\xi\right)\right]\right\} \\
&\times\exp\left\{-\mathrm{i}\kappa\int\mathrm{d}^4lh_{\mu\nu}^{\mathrm{ext}}(l)[J_1(l)\mathrm{e}^{\mathrm{i}lx_1}+j_2(l)\mathrm{e}^{\mathrm{i}lx_2}]\right\}. \quad (3.1)
\end{aligned}$$

Further we apply to $T(p_1,p_2;q_1,q_2|h^{\mathrm{ext}})$ the operator

$$\prod_{i=1}^N \frac{\epsilon_{\mu\nu}^i(k_i)}{(2\pi)^{3/2}\sqrt{2k_0 i}}\frac{\delta}{\delta h_{\mu\nu}^{\mathrm{ext}}(k_i)}; \quad (3.2)$$

then setting $h_{\mu\nu}^{\mathrm{ext}} = 0$, we obtained the amplitude for the production of N gravitons in the collision of two "nucleons":

$$(2\pi)^4\delta^4\left(p_1+p_2-q_1-q_2-\sum_{i=1}^{i=N}k_i\right)$$

$$
\begin{aligned}
&\times \mathrm{i}T_{\mathrm{inel}}(p_1,p_2;q_1,q_2|k_1,k_2,...k_N) \\
&= (\kappa^2)\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \mathrm{e}^{\mathrm{i}(p_1-q_1)x_1+\mathrm{i}(p_2-q_2)x_2} D^{\alpha\beta\gamma\delta}(x) \\
&\times \int [\delta^4\nu_1]_{-\infty}^{\infty} \int [\delta^4\nu_2]_{-\infty}^{\infty} [p_1+q_1+2\nu_1(0)]_\alpha \\
&\times [p_1+q_1+2\nu_1(0)]_\beta [p_2+q_2+2\nu_2(0)]_\gamma \\
&\times [p_2+q_2+2\nu_2(0)]_\delta] \\
&\times \int_0^1 \mathrm{d}\lambda \exp\left\{\frac{1}{2}\mathrm{i}\kappa^2\left[2\mathrm{i}\lambda \mathrm{e}^{\mathrm{i}kx}\int J_1DJ_2\right.\right. \\
&\times \left.\left.\sum_{i=1,2}\left(\int \mathrm{d}^4kJ_iDJ_i - \mathrm{i}\int_{-\infty}^{\infty}\delta_i m^2\mathrm{d}\xi\right)\right]\right\} \\
&\times \frac{1}{\sqrt{N!}}\prod_{i=1}^{N}\frac{\epsilon^i_{\mu\nu}(k_i)}{(2\pi)^{3/2}\sqrt{2k_0\mathrm{i}}}(-\mathrm{i}\kappa) \\
&\times [J_1(k_i)\mathrm{e}^{\mathrm{i}k_ix_1} + j_2(k_i)\mathrm{e}^{\mathrm{i}k_ix_2}], \qquad (3.3)
\end{aligned}
$$

where $\epsilon^i_{\mu\nu}(k_i)$ is the polarization tensor of a graviton with momentum k_i . We have introduced in (3.3) the factor $N!^{1/2}$ which takes into account the fact that the emitted gravitons are identical. In the approximation (2.8) the scattering amplitude of the inelastic process (2.16) takes the form

$$
\begin{aligned}
&T_{\mathrm{inel}}(p_1,p_2;q_1,q_2,k_1|k_2,...k_N) = \int \mathrm{d}^4x \mathrm{e}^{\mathrm{i}(p_1-q_1)x} \\
&\times [k+p_1+q_1]_\mu [k+p_1+q_1]_\nu D^{\mu\nu\rho\sigma}(x) \\
&\times [-k+p_2+q_2]_\rho [-k+p_2+q_2]_\sigma \\
&\times \int_0^1 \exp\left(-\frac{\mathrm{i}\lambda\kappa^2}{(2\pi)^4}\int \mathrm{d}^4kD_{\mu\nu\rho\sigma}(k)\mathrm{e}^{\mathrm{i}kx}\overline{J_1^{\mu\nu}(-k)J_2^{\rho\sigma}(k)}\right) \\
&\times \exp\left(-\frac{\kappa^2}{2}\sum_{i=1}^{2}\int (DJ_i - \delta m_i^2)\right) \\
&\times \frac{1}{\sqrt{N!}}\prod_{i=1}^{N}\frac{\epsilon^i_{\mu\nu}(k_i)}{(2\pi)^{3/2}\sqrt{2k_0\mathrm{i}}}(\mathrm{i}\kappa) \\
&\times [J_1^{\mu\nu}(k_i)\mathrm{e}^{\mathrm{i}k_ix/2} + J_2^{\mu\nu}(k_i)\mathrm{e}^{-\mathrm{i}k_ix/2}]. \qquad (3.4)
\end{aligned}
$$

In (3.4) we have taken into account the law of conservation of energy-momentum. We have separated out the δ^4-function $\delta^4\left(p_1+p_2-q_1-q_2-\sum_{i=1}^{i=N}k_i\right)$. Note that by virtue of our assumption (2.7) that the created gravitons have small momenta we can set $k_i = 0 (i = 1,2,3,...,N)$ in (3.4) in the expressions $\exp(\pm \mathrm{i}k_ix/2)$. In other words, we consider the production of "soft" gravitons which do not affect the motion of the scattered high energy "nucleons".

4 Asymptotic behavior of the differential cross section for multiple production

The differential cross section for the production of N gravitons in a collision of two "nucleons" is given by

$$
\begin{aligned}
\mathrm{d}\sigma_n &= \frac{1}{2\sqrt{s(s-4m^2)}}|T_{\mathrm{inel}}(p_1,p_2;q_1,q_2|k_1,k_2,...,k_N)|^2 \\
&\times \delta^4\left(p_1+p_2-q_1-q_2-\sum_{i=1}^{N}k_i\right) \\
&\times \frac{1}{(2\pi)^6}\frac{\mathrm{d}^3q_1}{2q_{10}}\frac{\mathrm{d}^3q_2}{2q_{20}}\cdot\frac{1}{n!}\prod_{i=1}^{n}\frac{\mathrm{d}^3k_i}{2k_{0i}}\frac{1}{(2\pi)^3}, \qquad (4.1)
\end{aligned}
$$

where $s = (p_1+p_2)^2$. In what follows we shall be interested in the asymptotic behavior of the differential cross sections for the production processes of "soft" gravitons whose momenta are restricted by the conditions (2.7). As we shall show we can neglect the interference terms in this case in the inelastic scattering amplitude (3.3) i.e.

$$
\begin{aligned}
&T_{\mathrm{inel}}(p_1,p_2;q_1,q_2,k_1,k_2,...k_N) = T_{\mathrm{el}}(p_1,p_2;q_1,q_2) \\
&\times \prod_{i=1}^{n_1}\epsilon^i_{\mu\nu}(k_i)\overline{J_1^{\mu\nu}(k,p_1,q_1)}\prod_{i=1}^{n_2}\epsilon^i_{\mu\nu}(k_i)\overline{J_2^{\mu\nu}(k,p_2,q_2)}, \qquad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
t = \Delta^2 &= \left(q_1 - p_1 + \sum_{i=1}^{i=n_1}k_i\right)^2 \\
&= \left(q_2 - p_2 + \sum_{i=1}^{i=n_2}k'_i\right)^2, \qquad (4.3)
\end{aligned}
$$

$$
n_1 + n_2 = N.
$$

Using (4.2) and the transformation

$$
\begin{aligned}
&\delta^4\left(p_1+p_2-q_1-q_2-\sum_{i=1}^{n_1}k_i-\sum_{i=1}^{n_2}k'_i\right) \\
&= \int \mathrm{d}^4\Delta\delta^4\left(p_1-q_1-\sum_{i=1}^{n_1}k_i+\Delta\right) \\
&\times \delta^4\left(p_2-q_2-\sum_{i=1}^{n_2}k'_i-\Delta\right), \qquad (4.4)
\end{aligned}
$$

we can represent the differential cross section for graviton production (4.1) in the form

$$
\begin{aligned}
(\mathrm{d}\sigma)_{n_1,n_2} &= \frac{1}{2s}\frac{\mathrm{d}^4\Delta}{(2\pi)^4}|T_{\mathrm{el}}(s,t)|^2 W_{n_1}(p_1,\Delta) \\
&\times W_{n_2}(p_2,-\Delta), \qquad (4.5)
\end{aligned}
$$

where $T_{\mathrm{el}}(p_1,p_2;q_1,q_2) = T_{\mathrm{el}}(s,t)$ is defined by (2.9), and

$$
\begin{aligned}
&W_{n_i}(p_i,\Delta) = \frac{2\pi}{n_i}\int \frac{\mathrm{d}^3q_i}{2q_{i0}}\delta^4\left(p_i-q_i-\sum_{i=1}k_i+\Delta\right) \\
&\times \prod_{i=1}^{n_i}\frac{\mathrm{d}^3k_i}{2k_{0i}}\frac{-\kappa^2}{(2\pi)^3}|\overline{J_i^{\mu\nu}(k,p_i,q_i)}|^2, \qquad (4.6)
\end{aligned}
$$

and there is a similar expression for the $W_{n_2}(p_2,-\Delta)$ The quantities $W_{n_1}(p_1,\Delta)$ and $W_{n_2}(p_2,-\Delta)$ depend on the variables

$$
t = \Delta^2, \quad r_1 = p_1\Delta,
$$

$$t = \Delta^2, \quad r_2 = p_2 \Delta, \tag{4.7}$$

respectively. Using the variables (4.7), we transform the volume element $\mathrm{d}^4\Delta$ to the form

$$\mathrm{d}^4\Delta = \frac{4\pi}{\sqrt{s(s-4m^2)}} \mathrm{d}t \mathrm{d}r_1 \mathrm{d}r_2 \frac{\mathrm{d}\phi}{2\pi}, \tag{4.8}$$

where ϕ is the azimuthal angle, the physical domain of the integration variables being given by the inequalities

$$-t \leq 2r_1 \leq s,$$

$$-t \leq 2r_2 \leq s,$$

$$s \geq m^2; \quad -s \leq t \leq 0. \tag{4.9}$$

In what follows we shall be interested in the differential cross section $(\mathrm{d}\sigma/\mathrm{d}t)_{n_1,n_2}$ in the limit $s \longrightarrow \infty$ with t fixed. Integrating (4.6) over $\mathrm{d}r_1$ and $\mathrm{d}r_2$ and using formula (2.17), and for t fixed, $|t| \ll m^2$; $s \longrightarrow \infty$, we obtain the expression

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_{n_1,n_2} \longrightarrow \frac{1}{4\pi}|f(t)|^2 \omega_{n_1}(s,t)\omega_{n_2}(s,t), \tag{4.10}$$

where

$$\begin{aligned}\omega_n(s,t) &= \frac{\mathrm{e}^{at}}{\pi} \int \mathrm{d}r W_n(s,t) \\ &= \frac{1}{n!}\mathrm{e}^{at} \prod_{i=1}^{n_i} \frac{\mathrm{d}^3k_i}{2k_{0i}} \frac{(-\kappa^2)}{(2\pi)^3} |\overline{J_i^{\mu\nu}(k,p_i,q_i)}|^2. \end{aligned} \tag{4.11}$$

The domain of integration Ω_p over the moment of the secondary gravitons is given by

$$-t \leq 2p \sum_{i=1}^{n} k_i - \left(\Delta - \sum_{i=1}^{n}\right)^2 \leq s, \tag{4.12}$$

or, since in our case $\left(\Delta - \sum_{i=1}^{n} k_i\right)^2 \approx \Delta^2$, by the condition

$$0 \leq 2p \sum_{i=1}^{n} k_i \leq s + t. \tag{4.13}$$

Let now consider the approximation in which one can neglect the total momentum of the emitted gravitons in accordance with the "softness" condition (2.7). In this approximation the expression (4.11) takes the form of a Poisson distribution,

$$\omega_n(s,t) = \frac{1}{n!}\mathrm{e}^{at}[\overline{n(s,t)}]^n, \tag{4.14}$$

where

$$\overline{n}(s,t) = -\frac{\kappa^2}{(2\pi)^3} \int \frac{\mathrm{d}^3k_i}{2k_{0i}} |\overline{J_i^{\mu\nu}(k,p_i,q_i)}|^2. \tag{4.15}$$

The integration (4.15) is effectively restricted by the conditions: $|k_z| \leq R_z$, $|k_z| \leq R_\perp$. The quantity $\overline{n}(s,t)$ play the role of the average number of particles in a collision of two "nucleons" at high energy $s \longrightarrow \infty$ and fixed t. In general, $\overline{n}(s,t)$ depends on the method chosen to cut off the integrals over the momenta of the emitted gravitons at the upper limit [23]. In particular, if

$$R_\perp^2 \sim m^2; \quad 1 \gg \alpha^2 \gg \mu^2/m^2,$$

$$\ln(m^2/\mu^2) \gg \ln(1/\alpha)^2; \quad \alpha = R_z/p_0, \tag{4.16}$$

$$|t| \leq m^2,$$

using formula (2.17) for $\overline{J_i^{\mu\nu}(k,p_i,q_i)}$, we find

$$\overline{n}(s,t) = -bt, \tag{4.17}$$

$$b = \frac{4Gm^2}{\pi}\left(\ln\frac{m^2}{\mu^2} + \frac{1}{2}\right), \tag{4.18}$$

which is twice the "nucleon" parameter (2.16) of the diffraction exponent function. Note also that the equation $2a = b$ holds in the infrared asymptotic limit $\mu \longrightarrow 0$. In this case the dependence on t cancels as a result of the summation in (4.11) over the number of all the emitted gravitons, and this leads to the disappearance of the diffraction peak in the differential cross section. A similar behavior was noted in [34] and is analogous to the self-similar behavior of the deep inelastic processes of the hadron interaction at high energy [33,34].

As we have mentioned, we have neglected the interference terms in the derivation (4.2); if we allowed for these terms in the exponent for $\overline{n}(s,t)$ we should obtain terms of the type

$$\frac{\kappa^2}{(2\pi)} \int \frac{\mathrm{d}^3k}{k_0} \overline{J_1^{\mu\nu}(-k,p_1,q_1)J_2^{\rho\sigma}(k,p_2,q_2)}, \tag{4.19}$$

which are infinitesimally small in the high energy limit $s \longrightarrow \infty$ with fixed t provided the conditions (4.16) above are satisfied [23].

5 Conclusions

In the framework of the functional integration method the asymptotic behavior of Planck energy elastic and inelastic amplitudes in quantum gravity is studied. A straight-line path approximation is used to calculate the functional integrals which arise. Closed relativistically invariant expressions are obtained for the two "nucleons" elastic and inelastic amplitudes including the radiative correction contributions. It is interesting to note that the total differential cross section summed over all the emitted gravitons may have no pronounced diffraction peak in a certain domain of momentum transfer. In this connection an analogy should be indicated with the automodel behavior of the cross section of high energy deep inelastic interactions of hadrons with leptons. Under the requirement of the "softness" of graviton production, the high energy two "nucleon" collision is considered by analogy with bremsstrahlung emission of soft particles in electrodynamics. The Poisson nature of the multiplicity distribution of secondary gravitons for fixed momentum transfers in high energy "nucleon" collisions is given.

The straight-line path approximation used in this work corresponds to a physical picture in which colliding high energy "nucleons" at the interaction receive a small recoil connected with the emission of "soft" gravitons and retain their individuality.

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